Computer Science 294 Lecture 3 Notes

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1 Influences, Effects, and Social Choice

1.1 Examples of decision rules

When studying social choice, we think of a boolean function $f : \{\pm 1\}^n \to \{\pm 1\}^n$ as a **voting rule**. We think of the input in $\{\pm 1\}^n$ as n votes and the output in $\{\pm 1\}$ as a collective decision. This corresponds to voting for an election between two candidates.

Example 1.1. For odd n, the **Majority function** $\mathrm{MAJ}_n : \{\pm 1\}^n \to \{\pm 1\}$ given by

$$\mathrm{MAJ}_n(x_1, \dots, x_n) = \mathrm{sgn}(x_1 + \cdots + x_n)$$

is one of the most natural voting rules.

Example 1.2. We can also require a unanimous vote to pass a decision. If we think of -1 as true and $+1$ as false, then this is the **And function** $\mathrm{AND}_n : \{\pm 1\}^n \to \{\pm 1\}$ given by

$$\mathrm{AND}_n(x_1, \dots, x_n) = \begin{cases} -1 & \text{if } x = (-1, \dots, -1) \\ +1 & \text{otherwise.} \end{cases}$$

Example 1.3. Alternatively, we can have the **Or function** $\mathrm{Or}_n : \{\pm 1\}^n \to \{\pm 1\}$ given by

$$\mathrm{Or}_n(x_1, \dots, x_n) = \begin{cases} +1 & \text{if } x = (+1, \dots, +1) \\ -1 & \text{otherwise.} \end{cases}$$

Example 1.4. The existence of a dictator corresponds to the character functions $\chi_{\{i\}} : \{\pm 1\}^n \to \{\pm 1\}$ which give $\chi_i(x) = x_i$.

We can generalize this idea of dictators to the situation where more than 1 person has power over the collective decision.

Definition 1.1. A function $f : \{\pm 1\}^n \to \{\pm 1\}$ is a k-**junta** if f "depends" on at most k of its input coordinates. Formally, there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $f(x) = g(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ for some boolean function $g : \{\pm 1\}^k \to \{\pm 1\}$.

Example 1.5. The Majority, And, and Or functions are not juntas, but the character $\chi_{\{i\}}$ is a 1-junta.

Definition 1.2. A function $f : \{\pm 1\}^n \to \{\pm 1\}$ is a **linear threshold function (LTF)** (or a **weighted majority**) if

$$f(x) = \operatorname{sgn}(a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n), \qquad a_0, a_1, \ldots, a_n \in \mathbb{R}.$$

The constant a_0 allows our decision to have an initial bias. All our examples so far are LTFs.

Example 1.6. The Or function is an LTF with $a_0 = -(n - 1/2)$ and $a_1 = a_2 = \cdots = 1$.

All monotone, symmetric functions are LTFs. This corresponds to the case where $a_1 = a_2 = \cdots = a_n = 1$.

Example 1.7. The **Tribes function** is the Or of s many tribes of size w; it is an Or of s Ands:

$$\mathrm{Tribes}_{w,s}(x) = \mathrm{Or}(\mathrm{And}(x_1, \ldots, x_w), \ldots, \mathrm{And}(x_{ws-w+1}, \ldots, x_{ws})).$$

The parameters s and w are referred to as the **size** and **width**, respectively. Usually, we pick $s \approx 2^w \cdot \ln 2$, and then

$$\mathbb{P}_{X \sim \{\pm 1\}^{ws}}(\mathrm{Tribes}_{w,s}(X) = \mathrm{True}) \approx \frac{1}{2}.$$

In general,

$$\mathbb{P}(\mathrm{Tribes}(X) = \mathrm{False}) = (1 - 2^{-w})^s.$$

1.2 Desirable properties of voting schemes

Here are some properties we may want our decision rules to have.

Definition 1.3. We say that f is **monotone** if for all vectors $x \le y$ (pointwise), $f(x) \le f(y)$.

Definition 1.4. We say that f is **symmetric** if $f(x) = f(x^\pi)$ for all $x \in \{\pm 1\}^n$ and permutations π, where $x^\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$.

Definition 1.5. We say that f is an **odd function** if $f(-x) = -f(x)$.

This corresponds to a kind of symmetry between the outcomes. Let's also introduce a weak form of symmetry.

Definition 1.6. Let $f : \{\pm 1\}^n \to \{\pm 1\}$. The **symmetry group** of f is $\mathrm{Aut}(f)$, the group of permutations $\pi \in S_n$ such that $f(x) = f(x^\pi)$ for all x.

The voting rule f is symmetric if and only if $\mathrm{Aut}(f) = S_n$.

Definition 1.7. We say that f is **transitive symmetric** if for all i, j, there exists a $\pi \in \mathrm{Aut}(f)$ such that $\pi(i) = j$.

Example 1.8. The Tribes function is transitive symmetric but not symmetric.

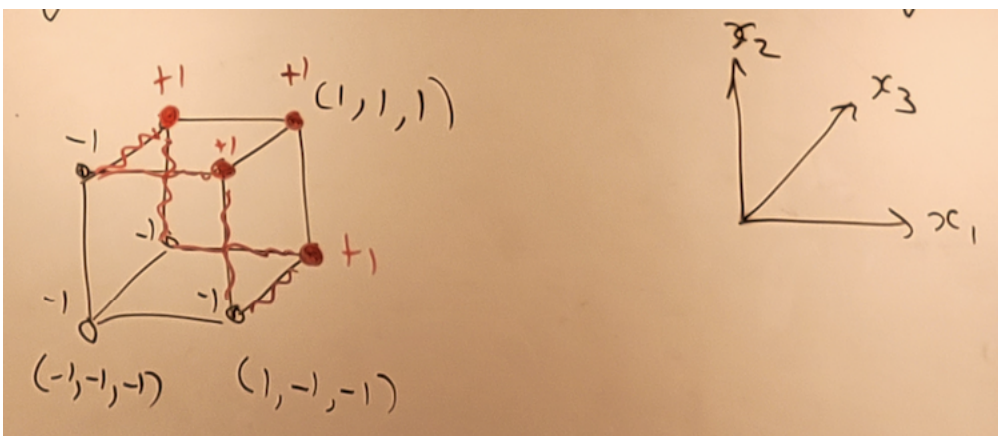
Definition 1.8. We say that f is **unanimous** if $f(b, \ldots, b) = b$ for all $b \in \{\pm 1\}$.

	Monotone	Symmetric	Odd	Transitive Symmetric	Unanimous
Tribes	✓			✓	✓
Maj	✓	✓	✓	✓	✓
Or	✓	✓		✓	✓
And	✓	✓		✓	✓
Dictatorship	✓		✓		✓

Remark 1.1. You can show that the Majority function is the only function which satisfies all these properties.

1.3 The influence of a decision rule

We can graph the Majority function on 3 variables:



Each edge corresponds to flipping only 1 bit in the input. The vertices where MAJ_3 outputs $+1$ and the edges that change the outcome are marked in red.

Definition 1.9. We say that coordinate i is **pivotal** (or **sensitive**) for $f : \{\pm 1\}^n \to \{\pm 1\}^n$ on input x if $f(x) \neq f(x^{\oplus i})$, where $x^{\oplus i} = (x_1, \ldots, x_{i+1}, -x_i, x_{i+1}, \ldots, x_n)$.

Definition 1.10. The **influence** of coordinate i on $f : \{\pm 1\}^n \to \{\pm 1\}$ is the probability that i is pivotal on a uniformly random input:

$$\mathrm{Inf}_i(d) = \mathbb{P}_{X \sim \{\pm 1\}^n}(f(X) \neq f(X^{\oplus i})).$$

We potentially want all these influences to be large so that voters feel that their vote matters.

Proposition 1.1.

$$\begin{aligned}\mathrm{Inf}_i(f) &= \frac{2 \cdot \#\ \textit{sensitive edges in direction } i}{\#\ \textit{vertices}} \\ &= \frac{\#\ \textit{sensitive edges in direction } i}{\#\ \textit{edges in direction } i}\end{aligned}$$

Example 1.9. In a dictatorship with $\chi_i(x) = x_i$, $\mathrm{Inf}_i(\chi_i) = 1$ and $\mathrm{Inf}_j(\chi_i) = 0$ for all $j \neq i$.

Example 1.10. For the Majority function with 3 voters, $\mathrm{Inf}_i(\mathrm{MAJ}_3) = \frac{2}{4} = \frac{1}{2}$ for all i.

Example 1.11. In general, for odd n,

$$\mathrm{Inf}_1(\mathrm{MAJ}_n) = \mathbb{P}_{X \sim \{\pm 1\}^n}(X_2 + \cdots + X_n = 0) = \frac{\binom{n-1}{(n-1)/2}}{2^{n-1}}.$$

By Stirling's formula, this looks like $\sqrt{2/\pi} \cdot 1/\sqrt{n}$ as $n \to \infty$.

1.4 The effect of a decision rule

Definition 1.11. The **effect** of coordinate i for $f : \{\pm 1\}^n \to \{\pm 1\}$ is

$$\mathrm{Eff}_i(f) = \mathbb{P}_{X \sim \{\pm 1\}^n}(f(X) = 1 \mid X_i = 1) - \mathbb{P}_{X \sim \{\pm 1\}^n}(f(X) = 1 \mid X_i = -1).$$

Remark 1.2. We have already shown that $\mathrm{Eff}_i(f) = \widehat{f}(\{i\}) = \langle f, \chi_i \rangle$.

Here is an example that illustrates the differences between effect and influence.

Example 1.12. The **Parity function** $\mathrm{Parity}_n : \{\pm 1\}^n \to \{\pm 1\}$ is given by

$$\mathrm{Parity}_n(x) = \prod_{i \in [n]} x_i.$$

The influence on f is $\mathrm{Inf}_i(f) = 1$ for each voter i. This voting rule maximizes influence for everyone! On the other hand, the effect of voter i on f is $\mathrm{Eff}_i(f) = 0$ for all i.

1.5 Derivatives of boolean functions

Next, we want to derive a nice Fourier representation for the influence, like we already have for the effect. Along the way, we will define a derivative.

Definition 1.12. The **derivative operator** D_i maps a function $f : \{\pm 1\}^n \to \mathbb{R}$ to the function $D_i f : \{\pm 1\}^n \to \mathbb{R}$ defined by

$$D_i f(x) = \frac{f(x^{(i \mapsto 1)}) - f(x^{(i \mapsto -1)})}{2},$$

where $x^{(i \mapsto \pm 1)} = (x_1, \dots, x_{i-1}, \pm 1, x_{i+1}, \dots, x_n)$.

Remark 1.3. Even though $D_i f$ ignores the i-th coordinate, we will still think of it as a function of n variables.

Note that

$$D_i f(x) = \begin{cases} \pm 1 & \text{if } i \text{ is pivotal on } x \\ 0 & \textbf{otherwise}, \end{cases}$$

so that

$$(D_i f(x))^2 = \mathbb{1}_{\{i\text{-th corrdinate is pivotal on } x\}}.$$

This lets us express the influence as

$$\mathrm{Inf}_i(f) = \mathbb{E}_X[(D_i f(X))^2].$$

Similarly,

$$\mathrm{Eff}_i(f) = \mathbb{E}_X[D_i f(X)].$$

Next time, we will use this to derive a Fourier representation for the influence.